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An engineering assessment to the relaxation time in thermal wave propagation

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Abstract—The physical significance of the relaxation time in the wave theory of heat conduction is further studied in this work. Thermodynamically, it is confirmed that the relaxation time results from the rate-equation within the mainframe of the second law in the nonequilibrium, irreversible thermodynamics. Mechanically, on the other hand, the relaxation time results from the phase-lag between the heat flux vector and the temperature gradient in a high-rate response. In transition from a diffusion behavior to the wave propagation, lastly, the relaxation time is found to be the physical instant at which the intrinsic length scales merge with each other.

INTRODUCTION

THE THERMAL wave theory is proposed to remove the paradox in the classical diffusion model assuming an infinite speed for heat propagation. The early work by Maxwell [1] on the kinetic theory of gases has had a significant impact on the recent development. The momentum transfer through collisions of molecules inspires the consideration of a finite speed of heat propagation in solids. The experiment made by Peshkov [2] directly shows that the thermal wave speed in liquid helium at 1.4 K is 19 m s^{-1}. Since then, the wave behavior in heat conduction has been argued from various physical points of view. It includes the hypothetical formulation suggested by Morse and Feshbach [3], Vernotte [4, 5], and Cattaneo [6]. The latter works by Chester [7] and Weymann [8] are representatives demonstrating the necessity of a finite speed of heat propagation from a microscopic point of view. These works, along with the arguments from the theory of relativity [9, 10], provides a physical foundation for the thermal wave phenomenon.

Continuing the works by Baumeister and Hamill [11], Taitel [12], and Wiggert [13], Özisik and his colleagues [14–16] remarkably advance the thermal wave theory to engineering applications. The discontinuity of temperature gradient at the thermal wavefront has attracted a lot of attention and presents a unique feature in the thermal wave propagation. The thermal shock formation around a fast-moving heat source [17–19] and in the vicinity of a rapidly propagating crack tip [20, 21] adds additional novelty into the thermal wave theory. When extended to predicting the crack trajectory emanating from the heat source [22, 23], the shock wave effect dramatically alters the fracture pattern predicted by the diffusion theory. For a crack dynamically propagating in 4340 steel, most importantly, the transonic and supersonic wave solutions obtained by Tzou [20] have presented several salient features in the experimental results obtained by Zehnder and Rosakis [24]. The encouraging coincidences between the theory and the experiment are reported in ref. [25]. It further confirms the value of thermal wave theory for engineering applications. Including all the other research efforts, the recent annual review article by Tzou [26] provides a complete source for additional references.

The relaxation behavior is the fundamental mechanism for the thermal resonance phenomenon [27–29] which cannot be depicted by diffusion. It bridges the thermal wave speed (C) to the thermal diffusivity (α) in the wave theory of heat conduction. Mathematically, the relaxation time τ is equal to α/C^2. Since the thermal diffusivity is well-tabulated, the value of τ relies on the thermal wave speed C. Generally speaking, the value of τ depends on temperature. For the thermal wave speed around 900 m s^{-1} in 4340 steel at 480°C [25], for example, the value of τ is of the order of 10^{-11} s. Due to the absence of a rigorous table for the thermal wave speed under various temperatures, a table for the relaxation time for engineering materials (including both fluid and gas) is, unfortunately, still absent at this point. Except some physical interpretations made from a microscopic point of view, moreover, the engineering significance of the relaxation time has not been fully addressed. The present work makes an attempt to examine the relaxation time in the thermal wave theory from the extended irreversible thermodynamics, the behavior of phase-lag in solids under high-rate response, and the transition of intrinsic length scale from the diffusion behavior to the wave propagation.

EXTENDED IRREVERSIBLE THERMODYNAMICS

Any admissible form of constitutive laws must satisfy the restrictions of thermodynamics. The classi-

NOMENCLATURE

A	positive-definite constant [W m^{-1} K]
a	speed of sound [m s^{-1}]
B	positive-definite constant [Pa s K]
C	thermal wave speed [m s^{-1}]
C_e	volumetric heat capacity of electron gas [J m^{-3} K^{-1}]
C_l	volumetric heat capacity of metal lattice [J m^{-3} K^{-1}]
C_p	heat capacity [J kg^{-1} K^{-1}]
C_1	integration constant, dimensionless
e	specific internal energy per unit per mass [J kg^{-1}]
G	electron–phonon coupling factor [W m^{-3} K^{-1}]
i, j	indices for coordinates, $i, j = 1, 2, 3$
J	entropy flux [W m^{-2} K^{-1}]
K	Boltzmann constant [J K^{-1}]
k	thermal conductivity [W m^{-1} k^{-1}]
$\mathbf{P}_{ij}$	pressure tensor [Pa]
p	thermodynamic pressure [Pa]
Q	rate of energy generation per unit volume [W m^{-3}]
$\mathbf{q}_i$	heat flux vector [W m^{-2}]
$\mathbf{r}_j$	position vector in Euclidean space [m]
s	specific entropy per unit mass [J kg^{-1} K^{-1}]
t	physical time [s]
T	absolute temperature [K]
$\mathbf{u}_i$	velocity vector [m s^{-1}]
v	specific volume [m^3 kg^{-1}]
x	one-dimensional space variable [m]
z	integrable variable [s].

Greek symbols

α	thermal diffusivity [m^2 s^{-1}]
β_T	isotropic constant [m s^2 J^{-1}]
β_M	isotropic constant [Pa^{-1}]
δ_{ij}	Kronecker delta function, dimensionless
ε_{ij}	deformation tensor [mm mm^{-1}]
η	intermediate time between 0 and t [s]
μ	fluid viscosity [Pa s]
ξ	variable of the similarity transformation, dimensionless
ρ	mass density [kg m^{-3}]
σ_{ij}	mechanical stress tensor [Pa]
Σ	entropy production rate per unit volume [W m^{-3} K^{-1}]
τ	relaxation time or phase lag of heat flux vector [s]
τ_M	relaxation time or phase lag of the stress tensor [s].

Other symbols

∇^2	Laplacian operator [m^{-2}].

Subscripts and superscripts

$X_{,i}$	$\partial X/\partial x_i$
$\dot{X}, X_{,t}$	$\partial X/\partial t$.

cal Fourier's law in heat conduction, for example, is an admissible form assuring a positive definite entropy production rate. As a brief review for this important character, let us consider the classical Gibb's equation which assumes local equilibrium for thermostatics [30–32]:

$$\mathrm{d}s = \frac{\mathrm{d}e}{T} + \frac{p\,\mathrm{d}v}{T}, \tag{1}$$

where s, e, v, T and p are, respectively, the specific entropy per unit mass, the specific internal energy per unit mass, the specific volume being reciprocal of the mass density ($v = 1/\rho$), the absolute temperature, and the thermodynamic pressure. The specific internal energy e and the specific volume (v) are state variables suitable for describing an equilibrium thermodynamic process, i.e. $s \equiv s(e, v)$. In relation to the rate change of entropy,

$$\left(\frac{\partial s}{\partial e}\right)_v = \frac{1}{T(e,v)}, \quad \left(\frac{\partial s}{\partial v}\right)_e = \frac{p(e,v)}{T(e,v)}. \tag{2}$$

The time-rate change of entropy per unit volume is thus

$$\rho\dot{s} = \frac{\rho\dot{e}}{T} + \frac{p\rho\dot{v}}{T}, \tag{3}$$

with superscript dots denoting differentiations with respect to time. Determination of $\rho\dot{e}$ and $\rho\dot{v}$ on the right-hand side of the equation lies in the conservation laws governing the physical processes of transition in the thermodynamic system [31, 32]:

$$\rho\dot{v} = \mathbf{u}_{i,i} \quad \text{(conservation of mass)} \tag{4}$$

$$\rho\dot{\mathbf{u}}_i = -\sigma_{ij,j} \quad \text{(conservation of momentum)} \tag{5}$$

$$\rho\dot{e} = -\mathbf{q}_{i,i} - \mathbf{P}_{ij}\dot{\varepsilon}_{ij} \quad \text{(conservation of energy)} \tag{6}$$

where homogeneous thermo–mechanical properties have been assumed. In these equations, repeated indices in a single term or products denote summations. The quantity $\mathbf{u}_i$ is the transport velocity in the medium, $\dot{\varepsilon}_{ij}$ the rate of deformation tensor defined as $\dot{\varepsilon}_{ij} = (\mathbf{u}_{i,j} + \mathbf{u}_{j,i})/2$, $\mathbf{P}_{ij}$ the pressure tensor, and $\mathbf{q}_i$ the heat flux vector. The pressure tensor ($\mathbf{P}_{ij}$) can be decomposed into the isotropic, thermodynamic pressure tensor ($p\delta_{ij}$) and the mechanical stress tensor (σ_{ij}). Mathematically,

$$\mathbf{P}_{ij} = p\delta_{ij} + \sigma_{ij}. \tag{7}$$

On substituting equations (4), (6), and (7) into (3), it results

$$\rho\dot{s}+\left(\frac{\mathbf{q}_i}{T}\right)_{,i} = -\frac{\mathbf{q}_i T_{,i}}{T^2}-\frac{\sigma_{ij}\dot{\varepsilon}_{ij}}{T}. \tag{8}$$

In comparison with the most general form of the conservation equation for entropy,

$$\rho\dot{s}+J_{i,i}=\Sigma, \tag{9}$$

with J being the entropy flux and Σ the entropy production rate per unit volume, it is clear that the entropy production rate Σ is composed of the products between the thermal and mechanical

$$J_i=\left(\frac{\mathbf{q}_i}{T}\right)_{,i},\quad \Sigma=\frac{\mathbf{q}_i T_{,i}}{T^2}-\frac{\sigma_{ij}\dot{\varepsilon}_{ij}}{T}, \tag{10}$$

fluxes ($\mathbf{q}_i$ and σ_{ij}, respectively) and the generalized driving forces ($T_{,i}/T^2$ and $\dot{\varepsilon}_{ij}/T$). The second law of thermodynamics requires Σ be positive-definite. In view of its quadratic form shown in (10), the *simplest* forms relating fluxes to driving forces are linear and represented by

$$\mathbf{q}_i=-\frac{A}{T^2}T_{,i}=-kT_{,i},\quad \sigma_{ij}=-\frac{B}{T}\dot{\varepsilon}_{ij}=-2\mu\dot{\varepsilon}_{ij},$$
$$\text{with } k>0,\ \mu>0. \tag{11}$$

They are some of the many admissible forms ensuring a positive-definite entropy production rate. Obviously, the first equation in (11) is the Fourier law in heat conduction while the second is the Stokes law of viscosity in fluid mechanics. They are linear approximations to general functionals satisfying the condition of $\Sigma > 0$. Replacing $\mathbf{q}_i$ in the first equation of (11) by $\mathbf{q}_i^3$, for example, also yields an admissible form ensuring $\Sigma > 0$. It yields, however, a nonlinear constitutive equation for heat conduction.

Note that the state variables, e and v, used in the Gibb's potential (1) or (3) are well-defined for thermodynamic processes assuming local equilibrium. When nonequilibrium transition occurs, additional state variables are needed to complete the description of the rate change of entropy. On a philosophical basis, the generalized flux is a natural consequence of an unbalanced thermodynamic driving force. It is ideal to be incorporated as a fundamental state variable for describing the nonequilibrium transition in a thermodynamic process [33]. The total change of entropy in a material volume is thus extended:

$$\mathrm{d}s=\left(\frac{\partial s}{\partial e}\right)\mathrm{d}e+\left(\frac{\partial s}{\partial v}\right)\mathrm{d}v+\left(\frac{\partial s}{\mathbf{q}_i}\right)\mathrm{d}\mathbf{q}_i+\left(\frac{\partial s}{\partial\sigma_{ij}}\right)\mathrm{d}\sigma_{ij}, \tag{12}$$

with $\mathbf{q}_i$ (the heat flux vector) and σ_{ij} (the stress tensor) serving as the generalized fluxes. In line with the classical thermodynamics assuming local equilibrium, the absolute temperature and pressure are well-defined:

$$\frac{1}{T}=\left(\frac{\partial s}{\partial e}\right)_{v,\mathbf{q}_i,\sigma_{ij}},\quad p=T\left(\frac{\partial s}{\partial v}\right)_{e,\mathbf{q}_i\sigma_{ij}}. \tag{13}$$

The associated derivatives with respect to $\mathbf{q}_i$ and σ_{ij},

$$\frac{\partial s}{\partial\mathbf{q}_i}=-\frac{v}{T}\beta_{\mathrm{T}}\mathbf{q}_i,\quad \frac{\partial s}{\partial\sigma_{ij}}=-\frac{v}{T}\beta_{\mathrm{M}}\sigma_{ij}, \tag{14}$$

are assumed for linear approximations perturbed from the local equilibrium state. The proportional constants β_{T} and β_{M} are scalar. The time-rate change of entropy, therefore, becomes

$$\rho\dot{s}=\frac{\rho\dot{e}}{T}+\frac{p\rho\dot{v}}{T}-\frac{\beta_{\mathrm{T}}}{T}\mathbf{q}_i\dot{\mathbf{q}}_i-\frac{\beta_{\mathrm{M}}}{T}\sigma_{ij}\dot{\sigma}_{ij}. \tag{15}$$

Substituting the conservation equations for mass and energy, (4) and (6), and using (7), it results

$$\rho\dot{s}+\left(\frac{\mathbf{q}_i}{T}\right)_{,i}$$
$$=\left[-\frac{\mathbf{q}_i T_{,i}}{T^2}-\frac{\beta_{\mathrm{T}}\mathbf{q}_i\dot{\mathbf{q}}_i}{T}\right]+\left[-\frac{\sigma_{ij}\dot{\varepsilon}_{ij}}{T}-\frac{\beta_{\mathrm{M}}\sigma_{ij}\dot{\sigma}_{ij}}{T}\right]. \tag{16}$$

While the entropy flux has the same form as that shown in (10), the entropy production rate in the present extended scheme becomes

$$\Sigma=\mathbf{q}_i\left[-\frac{T_{,i}}{T^2}-\frac{\beta_{\mathrm{T}}\dot{\mathbf{q}}_i}{T}\right]+\sigma_{ij}\left[-\frac{\dot{\varepsilon}_{ij}}{T}-\frac{\beta_{\mathrm{M}}\dot{\sigma}_{ij}}{T}\right]. \tag{17}$$

As required by the second law of thermodynamics, the entropy production rate must be positive definite. In view of the quadratic forms between the generalized fluxes and driving forces, similarly,

$$-\frac{T_{,i}}{T^2}-\frac{\beta_{\mathrm{T}}\dot{\mathbf{q}}_i}{T}=\frac{\mathbf{q}_i}{A},\quad -\frac{\dot{\varepsilon}_{ij}}{T}-\frac{\beta_{\mathrm{M}}\dot{\sigma}_{ij}}{T}=\frac{\sigma_{ij}}{B},$$
$$\text{with } A,\ B \text{ being positive} \tag{18}$$

are some of the many possibilities assuring $\Sigma > 0$. Rearranging equation (18), then, gives

$$\mathbf{q}_i+\tau\dot{\mathbf{q}}_i=-k\mathbf{T}_{,i},\quad \sigma_{ij}+\tau_{\mathrm{M}}\dot{\sigma}_{ij}=-2\mu\dot{\varepsilon}_{ij}, \tag{19}$$

where

$$k=\frac{A}{T^2},\quad 2\mu=\frac{B}{T},\quad \tau=\frac{A\beta_{\mathrm{T}}}{T},\quad \tau_{\mathrm{M}}=\frac{B\beta_{\mathrm{M}}}{T}. \tag{20}$$

For A and B being positive and temperature T measured on an absolute scale, $\Sigma > 0$ implies $k > 0$, $\mu > 0$, $\tau > 0$ and $\tau_{\mathrm{M}} > 0$. When $\tau = 0$ and $\tau_{\mathrm{M}} = 0$, the rate equations in equation (19) reduce to the constitutive laws in thermostatics, equation (11). While the second equation depicts the Maxwell flow behavior in non-Newtonian fluids [34], the first equation is indeed the constitutive law assumed in the linear theory of thermal wave propagation. Note also that τ and τ_{M} have the dimension of time. They are termed 'relaxation time' in the sense that even a zero driving force ($\mathbf{T}_{,i}$ or $\dot{\varepsilon}_{ij}$) results in an exponentially decayed response of flux ($\mathbf{q}_i$ or σ_{ij}) in time. In passing, it should be noted that the linear forms suggested by equation (19) are by no means rigorous. Should linear dependencies of $\partial s/\partial \mathbf{q}_i$ on σ_{ij} and $\partial s/\partial \sigma_{ij}$ on ξ_i be incorporated in equation (14), for example, a set of coupled constitutive equations would result. The approach by Jou *et al.* [33] provides a more general treatment involving mechanical work done by the deviatoric and the hydrostatic components. It also includes an updated survey about the development of rate-dependent constitutive equations in continuum mechanics.

THE PHASE LAG

Another approach to describe the relaxation behavior in thermal wave propagation is to employ the physical concept of time-lag between the heat flux vector and the temperature gradient

$$\mathbf{q}_i(\mathbf{r}_j,\ t+\tau) = -kT_i(\mathbf{r}_j,\ t), \tag{21}$$

where $\mathbf{r}_j$ is the position vector of the material volume under observation and k the principal thermal conductivity in the isotropic thermal conductivity tensor k_{ij}. Mathematically, $k_{ij} = k\delta_{ij}$. While the appearance of the Fourier law is retained, equation (21) depicts that the temperature gradient established at time t results in a delayed heat flux occurring at a later time $t+\tau$. Such a 'choking' phenomenon may be a mechanism of heat transfer in solids subject to thermal excitations varying with a high-rate. In the absence of a delayed response, $\tau = 0$, equation (21) reduces to the Fourier law in heat conduction. While equation (21) provides a general scheme to incorporate the physical mechanism of delayed responses, a linear version results from the Taylor's series expansion applied to $\mathbf{q}_i$ with respect to τ

$$\mathbf{q}_i(\mathbf{r}_j,\ t+\tau)$$
$$= \mathbf{q}_i(\mathbf{r}_j,\ t) + \frac{\mathrm{d}\mathbf{q}_i(\mathbf{r}_j,\ t)}{\mathrm{d}t}\tau + O(\tau^2) = -kT_{,i}(\mathbf{r}_j,\ t). \tag{22}$$

For τ being small, up to the first order, equation (22) is approximated by

$$\mathbf{q}_i(\mathbf{r}_j,\ t) + \tau\dot{\mathbf{q}}_i(\mathbf{r}_j,\ t) \cong -kT_{,i}(\mathbf{r}_j,\ t). \tag{23}$$

It has the same form as that in equation (19) derived from nonequilibrium thermodynamics. In addition to revealing the delayed response explicitly, equation (23) also relates the heat flux to the temperature gradient at the same instant of time. This is required for all the physical quantities involved in the constitutive laws, equations (4)–(6).

Though linearized, equation (23) does reveal an important characteristic, namely the path-dependency, in the thermal loading history. Integrating (23) directly, it yields

$$\mathbf{q}_i(\mathbf{r}_j,\ t) = -\frac{k}{\tau}\exp\left(-\frac{t}{\tau}\right)\int_0^t \exp\left(\frac{z}{\tau}\right)T_{,i}(\mathbf{r}_j,\ z)\,\mathrm{d}z. \tag{24}$$

In addition to the point value of $T_{,i}$ at time t, equation (24) indicates that the heat flux $\mathbf{q}_i$ also depends on the way in which the temperature gradient is accumulated from 0 to t. Apparently, the heat flux vector has a memory keeping track of the time-history of the temperature gradient. The integral relationship shown by (24) can be reduced to an algebraic expression by the mean value theorem,

$$\mathbf{q}_i(\mathbf{r}_j,\ t) = -\frac{kt}{\tau}\exp\left(\frac{\eta-t}{\tau}\right)T_{,i}(\mathbf{r}_j,\ \eta), \tag{25}$$

with η being an intermediate time between 0 and t. This is the reason that some researchers associate the fading memory to thermal wave propagation [35].

The intrinsic transition from a rate-independent (thermostatics) to a rate-dependent (thermodynamics) response is nontrivial. As a result of such a transition, mathematically, the energy equation (6) transits from a parabolic to a hyperbolic type. Discontinuities of temperature gradient at the wavefront and the thermal shock formation around a moving origin are typical examples resulting from such a transition.

INTRINSIC LENGTH SCALE

Mechanical work ($\mathbf{P}_{ij}\dot{\varepsilon}_{ij}$) was included in the energy equation (6) for the sake of generality. It may be neglected when emphasis is placed on the process of heat conduction. The temperature representation for the energy equation has been derived by many authors:

$$T_{,ii} + \frac{1}{\rho\alpha C_{\mathrm{p}}}(Q+\tau Q_{,t}) = \frac{1}{\alpha}T_{,t} + \frac{\tau}{\alpha}T_{,tt}, \tag{26}$$

where α is the thermal diffusivity and Q the body heat source. In the absence of time-lag, $\tau = 0$ and equation (26) reduces to the classical diffusion equation. Note that $T_{,ii} \equiv T_{,11} + T_{,22} + T_{,33} \equiv \nabla^2 T$ because the repeated index i is summed over. The presence of τ intrinsically alters the characteristics of temperature since it appears in the highest order time-derivative in the energy equation. Also, an apparent heat source, $(\tau/\rho\alpha C_{\mathrm{p}})Q_{,t}$, presents due to the relaxation behavior. It involves time-derivative of the real heat source generated from or applied to the solid medium [16, 17–19, 27–29].

By focusing attention on the terms of $T_{,ii}$ and $(\tau/\alpha)T_{,tt}$ in (26),

$$\frac{\partial^2 T}{\partial x_i\,\partial x_i} + \ldots = \ldots + \frac{1}{\left(\frac{\alpha}{\tau}\right)}\frac{\partial^2 T}{\partial t^2}, \tag{27}$$

it is evident that $(\alpha/\tau)t^2$ must have a dimension of square of a length. It suggests that the ratio $\sqrt{(\alpha/\tau)}$ must be a velocity-like quantity. In other words,

$$\frac{\alpha}{\tau} = C^2, \tag{28}$$

with C being the *thermal wave speed* in the solid medium. Both α and τ are intrinsic thermal properties of the medium. The resulting thermal wave speed C, therefore, is also an intrinsic thermal property. It characterizes the thermal wave propagation along with the diffusion behavior characterized by the diffusivity. This observation based on the dimensional consistency is by no means rigorous. However, it leads to the same result as that derived by the collision theory of molecules from a microscopic point of view [7, 8]. As termed by Chester [7], the reciprocal of the relaxation time, $f = 1/\tau$, is the critical frequency dictating the activation of thermal wave behavior. When the collision frequency among molecules exceeds such a threshold, the wave behavior in heat conduction dominates over diffusion.

A one-dimensional response without heat generation will be used to illustrate transition of the intrinsic length scale. For a relatively long-time response in transient, the diffusion behavior in heat conduction dominates over the wave behavior. In this case, the diffusion equation is retrieved:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \tag{29}$$

with x being the space variable. In a similar consideration for the dimensional consistency, it is evident that $\sqrt{(\alpha t)}$ has a dimension of length. The length is intrinsic in the sense that it involves the thermal diffusivity, an intrinsic property of the medium, and the response time. This observation, in essence, leads to the similarity transformation in the diffusion theory. Introducing

$$\xi = \frac{x}{\sqrt{(\alpha t)}} \tag{30}$$

as the dimensionless variable owing to the similar dimension of $\sqrt{(\alpha t)}$ to a length, the diffusion equation (29) is reduced to an ordinary differential equation,

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\xi^2} + \frac{\xi}{2}\frac{\mathrm{d}T}{\mathrm{d}\xi} = 0. \tag{31}$$

It yields the famous error-function-solution (for example [36]),

$$T(\xi) = C_1 \operatorname{erf}\left(\frac{\xi}{2}\right), \tag{32}$$

with constant C_1 determined from the appropriate initial/boundary conditions or heat source generated from the solid. The constant 2 normally involved in the similarity transformation, i.e. $\xi = x/\sqrt{(2\alpha t)}$, is to eliminate the 2 in equation (32). This is for the cleanness of the mathematical result and not absolutely necessary.

For a relatively short-time response, on the other hand, the wave behavior dominates over diffusion. Equation (26) in this case reduces to a wave equation,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 T}{\partial t^2}. \tag{33}$$

Obviously, the intrinsic length scale switches to Ct. The similarity transformation, consequently, is modified:

$$\xi = \frac{x}{Ct}. \tag{34}$$

Applying the chain rules for the differentiations in equation (33), then, the following ordinary differential equation results

$$(1-\xi^2)\frac{\mathrm{d}^2 T}{\mathrm{d}\xi^2} - 2\xi\frac{\mathrm{d}T}{\mathrm{d}\xi} = 0. \tag{35}$$

Solving for T, it gives

$$T(\xi) = C_1 \ln\left(\frac{1+\xi}{1-\xi}\right). \tag{36}$$

Due to the finite speed of heat propagation, no thermal disturbance is present in the physical domain of $x > Ct$ (ahead of the wavefront). The applicable range of equation (36) for temperature, therefore, is confined to the domain of $0 < \xi < 1$. In the vicinity of the wavefront at $x = Ct$, i.e. $\xi = 1$, a log-type singularity is present. Due to such a singularity in temperature, a pure wave behavior is not allowed in thermal wave propagation without the counterbalanced effects from diffusion. The Laplace transform solution obtained by Baumeister and Hamill [11] incorporates both effects of wave and diffusion. It has been shown that the temperature remains bounded at the thermal wavefront while the singularity of temperature gradient still presents.

Equations (30) and (34) provide the intrinsic length scales for the asymptotic behavior of thermal waves. In summary, the length scale for the long-time response in transient, namely diffusion, is $\sqrt{(\alpha t)}$ while that for the short-time response, namely wave, is Ct. On a physical basis, the two scales merge at a certain instant of time when both effects present. Equating (30) to (34), then, yields

$$t = \frac{\alpha}{C^2} \tag{37}$$

which has the same form as the relaxation time in equation (28). Because the thermal diffusivity and the thermal wave speed are intrinsic properties of the medium, the physical instant at which the intrinsic length scales in diffusion and wave merges is a thermal property of the medium. Different media have different merge times in transient.

CONCLUSION

The relaxation time in the thermal wave theory has been examined from various physical points of view. It is admissible within the mainframe of the second law in the nonequilibrium, irreversible thermodynamics. It is the time-lag between the heat flux vector and the temperature gradient when response time is short. It is also the physical instant at which the intrinsic length scale in diffusion transits to that in wave. These observations should provide sufficient physical basis to picture the relaxation time used for modelling the process in high-rate heat conduction.

In relation to the macroscopic thermal properties, the relaxation time τ is α/C^2. In a series of recent works correlating the thermal wave theory to the microscopic two-step model [37, 38], moreover, the thermal wave speed (C) is related to the coupling factor (G) of the electron–phonon collisions and the volumetric heat capacities of the electrons (C_e) and the metal lattice (C_l) by

$$C = \sqrt{\frac{kG}{C_e C_l}}. \tag{38}$$

The relaxation time (τ), consequently, is

$$\tau = \frac{1}{G\left(\frac{1}{C_e} + \frac{1}{C_l}\right)}. \tag{39}$$

The coupling factor of electron–phonon interactions depends on the number density of electrons (n_e), speed of sound (a), thermal conductivity (k), and Bolzmann constant (K). Mathematically, within the limit of Wiedemann–Franz's law [39], $G = \pi^4(n_e aK)^2/k$. In contrast to the macroscopic relationship, equation (39) expresses the relaxation time in terms of the microscopic quantities such as the number density of electrons and heat capacity of the electron gas. Through the speed of sound, moreover, the relaxation time further depends on the atomic number density, Debye temperature, and plank constant. The microscopic point of view may shed light on the determination of τ (the relaxation time) and C (the thermal wave speed) from solid state physics. Many quantities in equations (38) and (39) are temperature sensitive. The heat capacity of electron gas, for example, may linearly increase with temperature. For a temperature increase of 1500 K in the electron gas, which is typical for metals subjecting to a laser fluence of 50 J m^{-2} [39], the thermal wave speed may reduce to half (50%) of its value at room temperature. The relaxation time, consequently, may become five times larger. Such an estimate is still qualitative because the other physical properties are assumed constant. A detailed, quantitative analysis must accommodate the temperature dependence of all the physical properties as an entirety.

In developing a suitable model for describing certain phenomena in engineering, establishment of a rigorous physical basis and comparison with experimental observations are equally important. While the near-tip temperature field in the transonic and supersonic ranges [25] has been confirmed by a recent experiment [24] for dynamic crack propagation, the present work has addressed the physical meaning of the relaxation time in an engineering sense. It aims to enhance the physical foundation of the thermal wave theory and hopefully will lead to more research toward engineering applications of the wave theory in heat conduction.

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